Density matrix for the kink ground state of the ferromagnetic *XXZ* chain

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The exact expression for the density matrix of the kink ground state of the ferromagnetic *XXZ* chain is obtained. Utilizing this, we exactly calculate various correlation functions such as the longitudinal and transverse spin-spin correlation functions, and the ferromagnetic and antiferromagnetic string formation probabilities. The asymptotic behaviors of these correlation functions are also analyzed. As a consequence, we find that the spin-spin correlation functions decay exponentially for large distances, while the string formation probabilities exhibit Gaussian decay for large strings. We also evaluate the entanglement entropy, which shows interesting behaviors due to the lack of the translational invariance of the state.

I. INTRODUCTION

The effects of quantum fluctuations of interacting quantum systems can be investigated by studying the correlation functions or quantifying the entanglement of the system, which are currently paid much attention. The exact evaluation of the correlation functions, however, is still a challenging problem even when models are completely integrable. The spin-1/2 *XXZ* model in one dimension is one of the most fundamental models, which can be exactly solved by the Bethe ansatz. As concerns for the correlation functions in the antiferromagnetic ground state, a few exact results are known so far: several short distance spin-spin correlation functions (see Ref. [1], and references therein) and the ferromagnetic string formation probability (which is the probability to find a ferromagnetic string with certain length) for $\Delta=1/2$ (Δ: anisotropy parameter) [2,3].

In this paper, we intensively consider the correlation functions for the ferromagnetic regime of the *XXZ* chain. In this regime, it has been well-known that there are two translationally invariant ground states *up* and *down*, the state with all spins up and the state with all spins down. In addition to these trivial ground states, two classes of nontranslationally invariant ground states *kink* and *antikink* were found in Ref. [4] (see also Ref. [5] for finite *XXZ* chain with boundary magnetic field). Though it is not obvious that the kink is the ground state in the infinite lattice limit, the authors in Ref. [4] proved it under the assumption that the ground states should be "frustration free," i.e., minimize not only the energy of the total Hamiltonian, but also the energy of the local Hamiltonian. Furthermore, it was shown that the "frustration free" ground states up, down, kink, and antikink are the complete set of the ground states [6,7]. In Ref. [8], the exact value of the spectral gap was obtained and shown to be independent of the reference ground state.

More recently, from the interest in the correspondence between the ground state of the ferromagnetic *XXZ* chain and the "quantum" Hamiltonian of the crystal melting model, some special correlation functions such as the magnetization

and the longitudinal spin-spin correlation function of the kink ground state were exactly calculated [9]. Utilizing the generating function developed in Ref. [4], in this paper, we derive the exact expression of the density matrix for the kink ground state of the ferromagnetic *XXZ* chain. By using this, various correlation functions can be systematically calculated for arbitrary interaction strengths and for arbitrary distances. The following correlation functions are particularly calculated here: the transverse and longitudinal spin-spin correlation functions and the ferromagnetic (antiferromagnetic) string formation probabilities [10] which are the probabilities of finding a ferromagnetic (antiferromagnetic) string in the kink ground state. The entanglement entropy of the system is also evaluated (see Ref. [11] for a finite *XXZ* chain with boundary magnetic field). Analyzing the asymptotic behaviors of these correlation functions, we find that the spin-spin correlation functions exponentially decay for the large distances. On the other hand, both the ferromagnetic and the antiferromagnetic string formation probabilities exhibit Gaussian decay for large strings. Moreover, their exact amplitudes are systematically determined.

This paper is organized as follows. In the next section, the kink ground state of the infinite *XXZ* chain in the ferromagnetic regime is considered. By using the generating function, the exact expression for the density matrix is derived. From this, we concretely analyze various correlation functions in Sec. III. The asymptotic behavior of the correlation functions are discussed in Sec. IV. Section V is devoted to a conclusion.

II. DENSITY MATRIX OF THE KINK GROUND STATE

In this section, we derive the exact expression for the density matrix of the infinite ferromagnetic *XXZ* chain in the kink ground state. The Hamiltonian is defined by

$$H = - \sum_{m \in \mathbb{Z}} [\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)], \quad (1)$$

where σ_m^α, $\alpha=x,y,z$ are the Pauli matrices acting on the *m*th site and Δ is the anisotropy parameter. We shall consider the ferromagnetic regime $\Delta>1$. Parametrizing Δ as

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$$\Delta = \frac{q^{1/2} + q^{-1/2}}{2}, \qquad (2)$$

$\Delta > 1$ corresponds to $0 < q < 1$.

It is known that there are infinitely many zero energy kink ground states interpolating between spin up at $-\infty$ and spin down at ∞ [4,5]. A kink ground state is the superposition of kinks which have the same center. The center of the kink is defined as the half integer-valued position where the number of up spins on the right of it is equal to the number of down spins on the left of it. Denote the kink ground state whose center is at $j - \frac{1}{2}(j \in \mathbb{Z})$ by $|\Psi_j\rangle$. Any $|\Psi_j\rangle$ can be extracted from the following generating function [4]:

$$|\Psi(z)\rangle = \bigotimes_{x \in \mathbb{Z}_{<0}} (|\uparrow\rangle_x + z^{-1} q^{-1/2(1/2+x)} |\downarrow\rangle_x) \otimes \bigotimes_{y \in \mathbb{Z}_{\geqslant 0}} (|\downarrow\rangle_y$$
$$+ zq^{1/2(1/2+y)} |\uparrow\rangle_y). \qquad (3)$$

$|\Psi_j\rangle$ is the coefficient of z^j of the expansion of $|\Psi(z)\rangle$, i.e.,

$$|\Psi(z)\rangle = \sum_{j \in \mathbb{Z}} z^j |\Psi_j\rangle. \qquad (4)$$

Let us calculate the form factors

$$\left\langle \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} \right\rangle_l := \frac{\langle \Psi_k | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi_l \rangle}{\langle \Psi_k | \Psi_k \rangle^{1/2} \langle \Psi_l | \Psi_l \rangle^{1/2}}, \qquad (5)$$

where $\{\epsilon_j\}, \{\epsilon_j'\} \in \{+, -\}$ and $E_j^{\pm \pm} = (1 \pm \sigma_j^z)/2$, $E_j^{\pm \mp} = \sigma_j^{\pm} = (\sigma_j^x \pm i\sigma_j^y)/2$. x_j is the position of the site where the operator $E_{x_j}^{\epsilon_j' \epsilon_j}$ acts on, and is assumed to be $x_j \neq x_k$ for $j \neq k$. First we calculate the norm $\langle \Psi_n | \Psi_n \rangle$ appearing in the denominator of Eq. (5). It can be obtained by calculating $\langle \Psi(z) | \Psi(z) \rangle$:

$$\langle \Psi(z) | \Psi(z) \rangle = (-wq^{1/2}; q)_\infty (-w^{-1} q^{1/2}; q)_\infty$$
$$= \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^\infty w^j q^{j^2/2}, \qquad (6)$$

where $w = z^2$ and $(a;q)_\infty := \prod_{j=0}^\infty (1 - aq^j)$. In the second equality, we have used the Jacobi triple product identity

$$(q;q)_\infty (-wq^{1/2}; q)_\infty (-w^{-1} q^{1/2}; q)_\infty = \sum_{j=-\infty}^\infty w^j q^{j^2/2}. \qquad (7)$$

Noting $\langle \Psi(z) | \Psi(z) \rangle = \sum_{j=-\infty}^\infty w^j \langle \Psi_j | \Psi_j \rangle$, we have

$$\langle \Psi_j | \Psi_j \rangle = \frac{q^{j^2/2}}{(q;q)_\infty}. \qquad (8)$$

Next we compute $\langle \Psi_k | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi_l \rangle$ appearing in the numerator of Eq. (5). Note that $\langle \Psi_k | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi_l \rangle = 0$ unless $l - k = \delta$, where

$$\delta := \#\{j; (\epsilon_j, \epsilon_j') = (+, -)\} - \#\{j; (\epsilon_j, \epsilon_j') = (-, +)\}. \qquad (9)$$

Then one can see

$$\langle \Psi(z) | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi(z) \rangle = \sum_i w^{i/2} \langle \Psi_{(i-\delta)/2} | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi_{(i+\delta)/2} \rangle,$$

$$|\Psi_{j \notin \mathbb{Z}}\rangle := 0. \qquad (10)$$

By induction, one can show that the following holds:

$$\langle \Psi(z) | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi(z) \rangle$$
$$= \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^\infty w^j q^{j^2/2} \prod_{j=1}^n (w\zeta_j)^{[\epsilon_j' \epsilon_j]} \sum_{k=0}^\infty (-w)^k$$
$$\times \sum_{j=1}^n \frac{\zeta_j^{k+n-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)}, \qquad (11)$$

where $\zeta_j = q^{1/2+x_j}$ and $[++] = 1$, $[--] = 0$, $[+-] = [-+] = \frac{1}{2}$ [see the Appendix for the proof of Eq. (11)]. Comparing Eqs. (10) and (11), we obtain

$$\langle \Psi_{(i-\delta)/2} | \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} | \Psi_{(i+\delta)/2} \rangle$$
$$= \frac{1}{(q;q)_\infty} \prod_{j=1}^n (\zeta_j)^{[\epsilon_j' \epsilon_j]} \sum_{k=0}^\infty (-1)^k$$
$$\times \sum_{j=1}^n \frac{\zeta_j^{k+n-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)} q^{(i/2 - k - \sum_{m=1}^n [\epsilon_m' \epsilon_m])^2/2}. \qquad (12)$$

Combining Eqs. (5), (8), and (12), we finally arrive at

$$\left\langle \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} \right\rangle_{(i-\delta)/2 \quad (i+\delta)/2}$$
$$= \prod_{j=1}^n (\zeta_j)^{[\epsilon_j' \epsilon_j]} \sum_{k=0}^\infty (-1)^k$$
$$\times \sum_{j=1}^n \frac{\zeta_j^{k+n-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)} q^{(i/2 - k - \sum_{m=1}^n [\epsilon_m' \epsilon_m])^2/2 - (i^2 + \delta^2)/8}.$$

$$(13)$$

In particular, when the operator $\prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j}$ preserves the total spin, i.e., $\delta = 0$, and the center of the kink is at $-1/2$, i.e., $i = 0$, the density matrix elements of the kink ground state whose center is located at $-1/2$ is given by

$$\left\langle \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} \right\rangle := \left\langle \prod_{j=1}^n E_{x_j}^{\epsilon_j' \epsilon_j} \right\rangle_0$$
$$= \prod_{j=1}^n (\zeta_j)^{[\epsilon_j' \epsilon_j]} \times \sum_{k=0}^\infty (-1)^k$$
$$\times \sum_{j=1}^n \frac{\zeta_j^{k+n-1}}{\prod_{l \neq j} (\zeta_j - \zeta_l)} q^{(k + \sum_{m=1}^n [\epsilon_m' \epsilon_m])^2/2}. \qquad (14)$$

III. CORRELATION FUNCTIONS

Here we analyze several crucial correlation functions: the magnetization, the longitudinal and transverse spin-spin correlation functions, and the ferromagnetic (antiferromagnetic) string formation probability which is the probability finding a ferromagnetic (antiferromagnetic) string in the kink ground state. At the end of this section, the entanglement entropy of the system is also considered. These correlation functions are directly calculated by the density matrix (14) derived in the preceding section. Note that the correlation functions discussed here are for the kink ground state whose center is located at $-1/2$. Other cases can also be treated by using Eq. (13). Let us list the explicit expressions of these correlation functions.

(i) Magnetization [9]:

$$\langle \sigma_x^z \rangle = 1 - 2\sum_{k=0}^{\infty} (-1)^k q^{k^2/2} q^{k(x+1/2)}. \tag{15}$$

(ii) Longitudinal spin-spin correlation function [9]:

$$\langle \sigma_{x_1}^z \sigma_{x_2}^z \rangle = 1 + 2\sum_{k=0}^{\infty} (-1)^{k+1} q^{(k+1)^2/2}$$
$$\times \frac{(q^{x_1+1/2} + q^{x_2+1/2})(q^{(k+1)(x_1+1/2)} - q^{(k+1)(x_2+1/2)})}{q^{x_1+1/2} - q^{x_2+1/2}}. \tag{16}$$

(iii) Transverse spin-spin correlation function

$$\langle \sigma_{x_1}^+ \sigma_{x_2}^- \rangle = q^{(1/2)(x_1+1/2)} q^{(1/2)(x_2+1/2)} \sum_{k=0}^{\infty} (-1)^k q^{(k+1)^2/2}$$
$$\times \frac{q^{(k+1)(x_1+1/2)} - q^{(k+1)(x_2+1/2)}}{q^{x_1+1/2} - q^{x_2+1/2}}. \tag{17}$$

(iv) Ferromagnetic string formation probability:

$$P_f(x,n) := \langle E_x^{++} \cdots E_{x+n-1}^{++} \rangle$$
$$= q^{nx+n^2/2} \sum_{k=0}^{\infty} (-1)^k q^{(k+n)^2/2} \sum_{j=1}^{n} \frac{q^{(x+j-1/2)k}}{\prod_{l \neq j} (1 - q^{l-j})}. \tag{18}$$

(v) Antiferromagnetic string formation probability:

$$P_a(x,n) := \langle E_x^{++} E_{x+1}^{--} \cdots E_{x+n-1}^{\pm\pm} \rangle + \langle E_x^{--} E_{x+1}^{++} \cdots E_{x+n-1}^{\mp\mp} \rangle = \sum_{k=0}^{\infty} (-1)^k \sum_{j=1}^{n} \frac{q^{(x+j-1/2)k}}{\prod_{l \neq j} (1 - q^{l-j})} q^{(n/2)(x+n/2)}$$

$$\times \begin{cases} (q^{(x+n/2)/2 + [k+(n+1)/2]^2/2} + q^{(-x-n/2)/2 + [k+(n-1)/2]^2/2}) & n : \text{odd}, \\ (q^{-n/2+(k+n/2)^2/2} + q^{n/2+(k+n/2)^2/2}) & n : \text{even} \end{cases} \tag{19}$$

Figure 1 shows the magnetization $\langle \sigma_x^z \rangle$. One sees that the sign of the magnetization changes between $x=-1$ and $x=0$, which corresponds to the fact that the center of the kink is located at $x=-1/2$. One can also see that most of the spins are aligned up for $x \ll -1$ and aligned down for $x \gg 1$. As decreasing of q (or equivalently as increasing of the aniso-
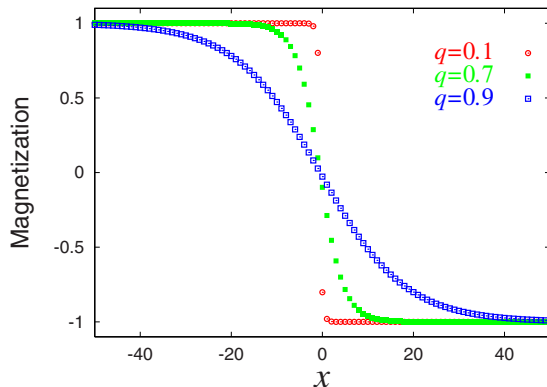

FIG. 1. (Color online) Magnetization $\langle \sigma_x^z \rangle$.

tropy parameter Δ), the slope of the magnetization curve around $x=0$ increases, and eventually will be infinite at the Ising limit $q=0$ ($\Delta=\infty$).

The longitudinal and the transverse spin-spin correlation functions

$$C^{zz}(x_1,x) := \langle \sigma_{x_1}^z \sigma_x^z \rangle - \langle \sigma_{x_1}^z \rangle \langle \sigma_x^z \rangle \tag{20}$$

and $\langle \sigma_{x_1}^+ \sigma_x^- \rangle$ are depicted in Fig. 2 for the case $x_1=-10$ and various anisotropies. Both the correlation functions have a peak around $x=0$. This characteristic behavior reflects the fact that the sign of the magnetization changes around $x=0$ and almost all spins are aligned at $|x| \gg 1$. These correlation functions decay exponentially for $|x| \gg 1$ (see Sec. III).

The ferromagnetic (antiferromagnetic) string formation probability $P_f(x,n)$ [$P_a(x,n)$] is the probability that the spins located in the region $[x,x+n-1]$ form a ferromagnetic (antiferromagnetic) string. These correlation functions are depicted in Figs. 3 and 4, respectively. From Figs. 3(b) and 4(b), one can see $P_f(x,m) > P_f(x,n)$ and $P_a(x,m) > P_a(x,n)$ for $m < n$, as expected. As shown in Sec. IV, both the string
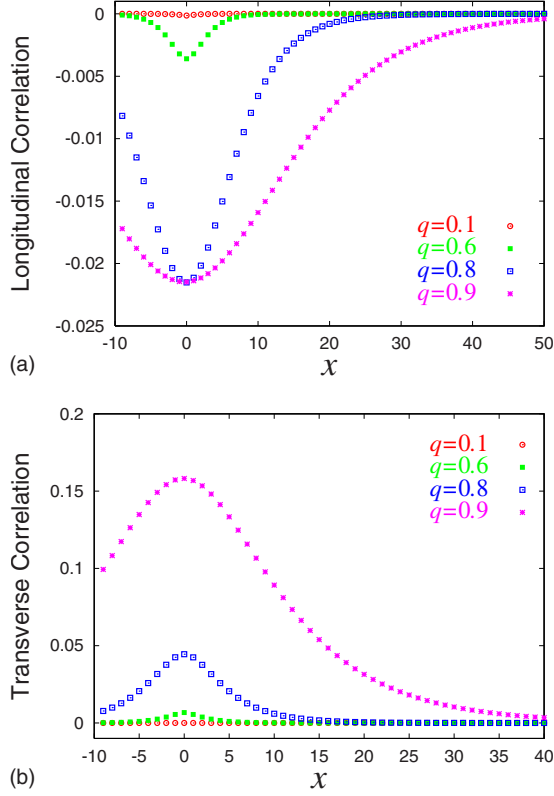
FIG. 2. (Color online) Top: The longitudinal spin-spin correlation functions $C^{zz}(x_1, x) := \langle \sigma_{x_1}^z \sigma_x^z \rangle - \langle \sigma_{x_1}^z \rangle \langle \sigma_x^z \rangle$ for $x_1 = -10$ and $-10 < x \leqslant 50$. Bottom: The transverse correlation functions $\langle \sigma_{x_1}^+ \sigma_x^- \rangle$ for $x_1 = -10$ and $-10 < x \leqslant 40$.

FIG. 3. (Color online) The ferromagnetic string formation probability (FSFP) $P_{\mathrm{f}}(x, n)$ for $n = 4$ and various anisotropies q (a), and for $q = 0.9$ and various lengths n (b).

formation probabilities $P_f(x, n)$ and $P_a(x, n)$ exhibit Gaussian decay for large strings $n \gg 1$. As the anisotropy parameter becomes larger, the effect of Ising interaction becomes stronger than that of quantum fluctuation. In the limit $q \to 0$ ($\Delta \to \infty$), the spins for $x < 0$ and $x > 0$ are all aligned up and down, respectively (see Fig. 1). This is reflected in the slope in Fig. 3(a) becoming steeper, and the peak in Fig. 4(a) becoming sharper, as the anisotropy parameter becomes larger.

One can also calculate the entanglement entropy $S(x, n)$, the von Neumann entropy of a subsystem $[x, x+1, \ldots, x+n-1]$. It is defined as

$$S(x, n) = -\mathrm{tr}\rho(x, n)\log_2 \rho(x, n), \tag{21}$$

where $\rho(x, n)$ is the reduced density matrix defined by tracing out the degrees of freedom of the environment outside the subsystem $[x, x+1, \ldots, x+n-1]$:

$$\rho(x, n) = \mathrm{tr}_E |\Psi_0\rangle\langle\Psi_0| = [P_{\epsilon_1, \ldots, \epsilon_n}^{\epsilon_1', \ldots, \epsilon_n'}(x, n)]_{\epsilon_j, \epsilon_j' = \pm}, \tag{22}$$

where

$$P_{\epsilon_1, \ldots, \epsilon_n}^{\epsilon_1', \ldots, \epsilon_n'}(x, n) = \left\langle \prod_{j=1}^{n} E_{x+j-1}^{\epsilon_j' \epsilon_j} \right\rangle. \tag{23}$$

Shown in Fig. 5 is the entanglement entropy. As $x \to \pm\infty$, the entanglement entropy of the kink ground state is asymptotically 0, which is nothing but that of the ferromagnetic ground state up and down. In Fig. 5(a), we observe an in-
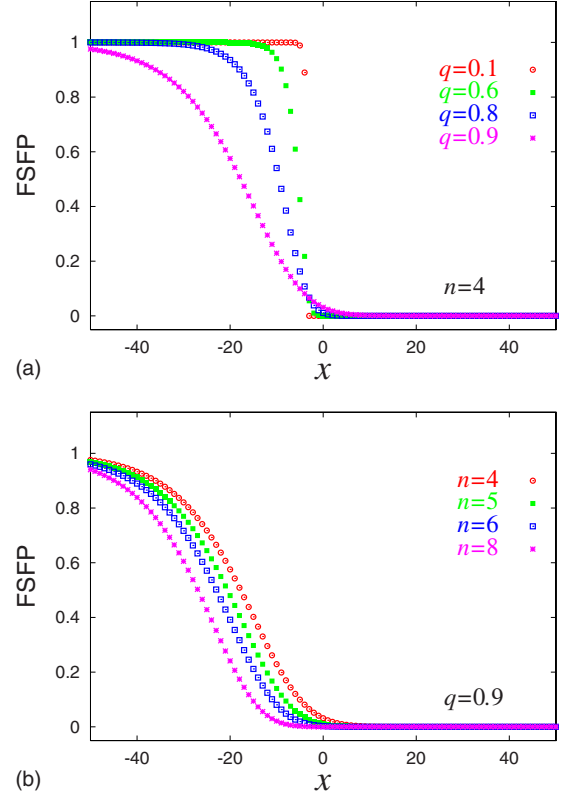
triguing phenomena that the peak of the entanglement entropy splits into two, as decreasing the parameter q (or equivalently as increasing the anisotropy parameter Δ). On the other hand, for fixed q, the same behavior can also be observed in Fig. 5(b) as increasing the length n of the sub-chain.

IV. ASYMPTOTICS

In this section, the asymptotic behaviors of the correlation functions derived in the preceding section are analyzed. Let us first consider the spin-spin correlation functions. From Eqs. (15) and (16), we find

$$\langle \sigma_{x_2}^z \rangle \xrightarrow{x_2 \to +\infty} -1 + 2q^{x_2+1},$$

$$\langle \sigma_{x_1}^z \sigma_{x_2}^z \rangle \xrightarrow{x_2 \to +\infty} -\langle \sigma_{x_1}^z \rangle$$

$$+ \left(2 + 4\sum_{k=0}^{\infty} (-1)^{k+1} q^{k^2/2 + k(x_1+3/2)}\right) q^{x_2+1}. \tag{24}$$

Thus we obtain

$$\langle \sigma_{x_1}^z \sigma_{x_2}^z \rangle - \langle \sigma_{x_1}^z \rangle \langle \sigma_{x_2}^z \rangle \sim A^{zz}(x_1) q^{x_2+1} \quad \text{for } x_2 \gg 1,$$
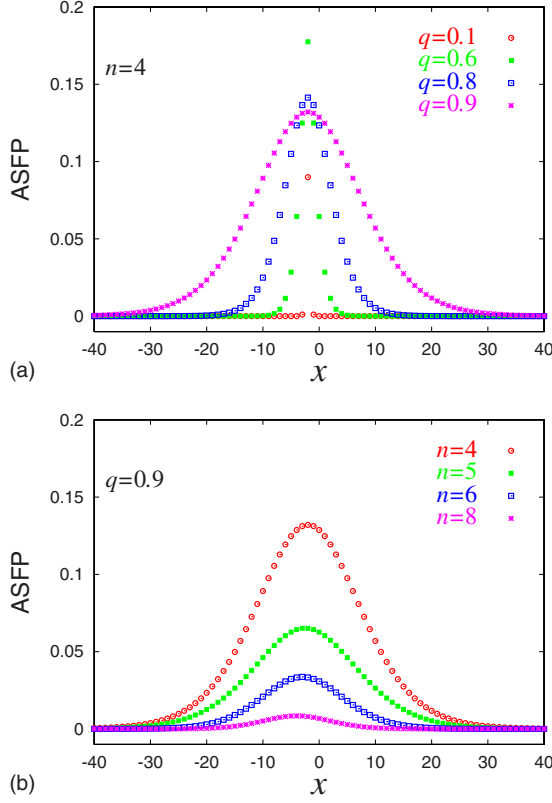
FIG. 4. (Color online) The antiferromagnetic string formation probability (ASFP) $P_a(x, n)$ for $n = 4$ and various anisotropies q (a) and for $q = 0.9$ and various lengths n (b).
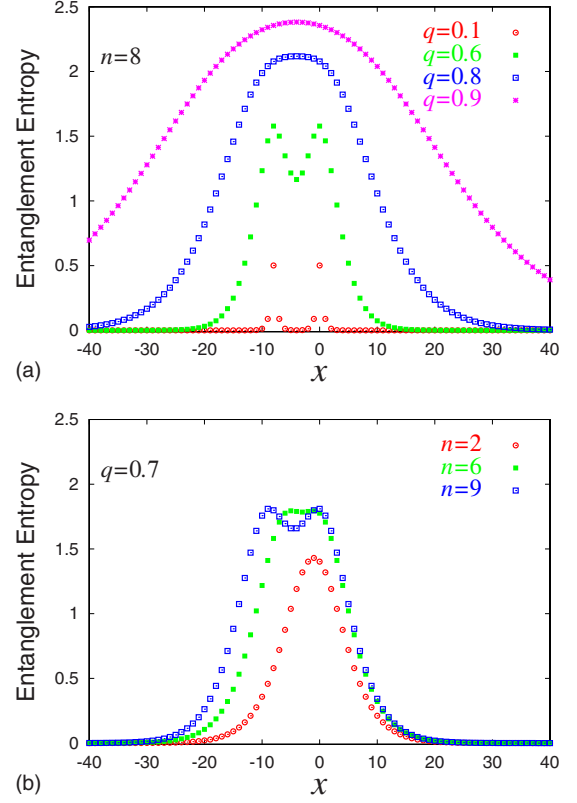

FIG. 5. (Color online) The entanglement entropy $S(x, n)$ for $n = 8$ and various anisotropies q (a) and for $q = 0.7$ and various lengths n (b).

$$A^{zz}(x_1) = 4\sum_{k=0}^{\infty} (-1)^k (1 - q^k) q^{k^2/2 + k(x_1 + 1/2)}. \quad (25)$$

This shows that the longitudinal spin-spin correlation function decays exponentially. The asymptotics of the transverse spin-spin correlation function (17) is also evaluated in the same manner:

$$\langle \sigma_{x_1}^+ \sigma_{x_2}^- \rangle \sim A^{+-}(x_1) q^{1/2(x_2 + 1/2)} \quad \text{for } x_2 \gg 1,$$

$$A^{+-}(x_1) = \sum_{k=0}^{\infty} (-1)^k q^{(k+1)^2/2 + (k+1/2)(x_1 + 1/2)}, \quad (26)$$

which shows that the transverse spin-spin correlation function also exhibits exponential decay.

Now, let us analyze the asymptotics of the string formation probabilities (18). Using the identity

$$\sum_{j=1}^{n} \frac{1}{\prod_{\substack{l \neq j}}^{n} (1 - q^{l-j})} = 1, \quad (27)$$

$P_f(x, n)$ can be rewritten as

$$P_f(x, n) = q^{nx + n^2}[1 + B(n)],$$

$$B(n) = \sum_{k=1}^{\infty} (-1)^k q^{k^2/2 + (n+x-1/2)k} \sum_{j=1}^{n} \frac{q^{jk}}{\prod_{\substack{l \neq j}}^{n} (1 - q^{l-j})}. \quad (28)$$

Since

$$|B(n)| < \sum_{k=1}^{\infty} q^{k^2/2 + (n+x-1/2)k} \sum_{j=1}^{n} \frac{q^{jk}}{\prod_{\substack{l \neq j}}^{n} (1 - q^{l-j})}$$

$$< \sum_{k=1}^{\infty} q^{(n+x-1/2)k} \sum_{j=1}^{n} \frac{1}{\prod_{\substack{l \neq j}}^{n} (1 - q^{l-j})} = \sum_{k=1}^{\infty} q^{(n+x-1/2)k}, \quad (29)$$

then

$$P_f(x, n) \sim q^{nx + n^2} \quad \text{for } n \gg 1. \quad (30)$$

This means that the ferromagnetic string formation probability shows Gaussian decay for large strings, which was indicated in Ref. [12]. Note here that similar Gaussian behaviors are also seen in the antiferromagnetic ground state [3,13,14].

Finally we explicitly write down the asymptotics of the antiferromagnetic string formation probability (19):

$$P_a(x,n) \sim q^{nx/2+3n^2/8}$$
$$\times \begin{cases} (q^{x/2+(4n+1)/8} + q^{-x/2+(-4n+1)/8}), & n:\text{odd}, \\ (q^{-n/8} + q^{n/8}), & n:\text{even} \end{cases}$$
$$(31)$$

for $n \geqslant 1$.

V. CONCLUSION

In this paper, the density matrix in the kink ground state of the ferromagnetic spin-$1/2$ *XXZ* chain has been exactly calculated. From this expression, the longitudinal and transverse spin-spin correlation functions, and the ferromagnetic and the antiferromagnetic string formation probability for arbitrary distances and arbitrary interaction strengths have been systematically calculated. Analyzing them, we find that the spin-spin correlation functions decay exponentially for large distances, while the string formation probabilities show Gaussian decay for large strings. We have also calculated the entanglement entropy and observed the change of shape with the increase of the anisotropy parameter or the length of the subchain.

There are several directions to which the systematic study of the correlation functions of the kink ground state may be generalized. The extension of the analysis to higher spins should be a straightforward exercise. The kink ground states can also be generalized to higher dimensions as interface states. Another type of generalization is to a droplet state which has a different topological structure, formed by combining a kink and antikink state. It is interesting to extend the analysis to these states (see Refs. [15–17] for studies on these states).

ACKNOWLEDGMENTS

APPENDIX: PROOF OF EQ. (11)

Let us show Eq. (11). Using Eq. (3), one obtains

$$\langle \Psi(z) | \prod_{j=1}^{n} E_{x_j}^{\epsilon'_j \epsilon_j} | \Psi(z) \rangle$$
$$= \prod_{j=1}^{n} \frac{1}{1+w\zeta_j} \prod_{j=1}^{n} (w\zeta_j)^{[\epsilon'_j \epsilon_j]} (-wq^{1/2};q)_\infty (-w^{-1}q^{1/2};q)_\infty$$
$$= g_w^n(\zeta_1, \ldots, \zeta_n) \prod_{j=1}^{n} (w\zeta_j)^{[\epsilon'_j \epsilon_j]} \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} w^j q^{j^2/2},$$

$$g_w^n(\zeta_1, \ldots, \zeta_n) := \prod_{j=1}^{n} \frac{1}{1+w\zeta_j}. \qquad (A1)$$

Expressing the Laurent expansion of $g_w^n(\zeta_1, \ldots, \zeta_n)$ as

$$g_w^n(\zeta_1, \ldots, \zeta_n) = \sum_{j=0}^{\infty} w^j \lambda_j^{(n)}(\zeta_1, \ldots, \zeta_n), \qquad (A2)$$

we find $\lambda_j^{(n)}(\zeta_1, \ldots, \zeta_n)$ satisfies the following recursion relation:

$$\lambda_k^{(n+1)}(\zeta_1, \ldots, \zeta_{n+1}) = \sum_{j=0}^{k} (-1)^{k-j} \zeta_{n+1}^{k-j} \lambda_j^{(n)}(\zeta_1, \ldots, \zeta_n). \qquad (A3)$$

To prove Eq. (11) is to show that $\lambda_j^{(n)}(\zeta_1, \ldots, \zeta_n)$ is

$$\lambda_j^{(n)}(\zeta_1, \ldots, \zeta_n) = (-1)^j \sum_{l=1}^{n} \frac{\zeta_l^{j+n-1}}{\prod_{i \neq l}(\zeta_l - \zeta_i)}. \qquad (A4)$$

Let us show this by induction. It is obvious that Eq. (A4) holds for $n=1$. Suppose it holds for n. Then from Eq. (A3), $\lambda_k^{(n+1)}(\zeta_1, \ldots, \zeta_{n+1})$ can be calculated as follows:

$$\lambda_k^{(n+1)}(\zeta_1, \ldots, \zeta_{n+1}) = (-1)^k \zeta_{n+1}^k \sum_{j=0}^{k} \sum_{l=1}^{n} \zeta_l^{n-1} \left(\frac{\zeta_l}{\zeta_{n+1}}\right)^j \frac{1}{\prod_{\substack{i=1 \\ i \neq l}}^{n}(\zeta_l - \zeta_i)}$$

$$= (-1)^k \sum_{l=1}^{n} \frac{\zeta_l^{n+k} - \zeta_l^{n-1}\zeta_{n+1}^{k+1}}{\prod_{\substack{i=1 \\ i \neq l}}^{n+1}(\zeta_l - \zeta_i)}$$

$$= (-1)^k \sum_{l=1}^{n+1} \frac{\zeta_l^{n+k}}{\prod_{\substack{i=1 \\ i \neq l}}^{n+1}(\zeta_l - \zeta_i)}. \qquad (A5)$$

In the last equality, we used

$$\sum_{j=1}^{n} \frac{\zeta_j^{n-2}}{\prod_{k \neq j}(\zeta_j - \zeta_k)} = 0 \quad \text{for } n \geqslant 2. \qquad (A6)$$

From Eq. (A5), we can see Eq. (A4) holds for $n+1$, which means Eq. (A4) holds for any n. Thus, from Eqs. (A1), (A2), and (A4), we obtain Eq. (11).

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